NUMERICAL EXPERIMENTS ON MASS LUMPING FOR THE ADVECTION-DIFFUSION EQUATION

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Abstract

Mass lumping is a numerical technique related to the finite element method (FEM) that has been widely used in different applications, although there is no agreement concerning this technique. In this work a review on publications related to mass lumping is presented. For comparison of both schemes, a numerical experiment is performed for the two-dimensional advection-diffusion equation. Both consistent and lumped mass matrix integration are compared in terms of accuracy and convergence rate for an iterative solver. The results obtained indicate that the consistent mass integration leads to more stable solutions.

Key words: mass lumping, numerical diffusion, oscillation, convergence.

Introduction

Physical processes can be investigated both in time and space, when the natural problem can be described mathematically by partial differential equations. Considering irregular domain boundaries or heterogeneities the solution of the proposed problem can only be achieved by use of discrete numerical methods. In this case, the approximation generally leads to system of equations of the type

$$M\dot{u} + Bu = P$$

in which *u* is the time dependent unknown variable vector, M is a mass (or accumulation) matrix, B is a stiffness matrix and P is the vector of loads.

In the case of the finite element method (FEM), the mass matrix can be handled either consistently or by mass lumping. Mass lumping is a numerical technique, which attempts to transmit a feature of the finite differences method (FDM) to the FEM.

Approximating a differential equation by means of the FEM, when the accumulation term is consistently integrated, results from the expression: [Please refer to Zienkiewicz & Morgan (1983) for details on the method]

$$\mathbf{M}^c = S \int_{\Omega^e} N_i N_j d\Omega$$

where S is a physical storage coefficient, N is the interpolation function and Ω is the element domain. Consequently, the mass matrix is non-diagonal and the solution of the system of equations is always implicit and numeric expensive, either in case of explicit time approximation. For a rectangular element, the mass matrix has then the form

$$\mathbf{M}^c = \frac{S\Delta x^2}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ & 4 & 2 & 1 \\ & & 4 & 2 \\ & & & 4 \end{bmatrix}$$

Summing all line coefficients onto the diagonal, the mass matrix is transformed into a diagonal form; through what the higher rate of the explicit time approximation can be exploited. This procedure is referred as mass lumping. In the literature the designation CVFE (Control Volume Finite Element) can also commonly be found.

$$m_{ij}^l = \delta_{ij} \sum_j m_{ij}^c$$

In terms of FEM it means that the mass matrix is integrated using the expression

$$\mathbf{M}^{l} = S \int_{\Omega^e} N_i d\Omega$$

For the same rectangular element used for the consistent integration, the mass matrix results in

$$\mathbf{M}^{l} = \frac{S\Delta x^2}{36}\begin{bmatrix} 9 & & & \\ & 9 & & \\ & & 9 & \\ & & & 9 \end{bmatrix}$$

Physically it means that, using mass lumping, the storage capacity of the field is no more distributed over an element, as in the case of consistent FEM (Figure 1a), but is limited to the mesh nodes (Figure 1b). In the case of the methods of finite differences (FDM) and finite volumes (FVM), this is a natural feature. For the finite element method (FEM), this feature is achieved artificially.

The same effect is achieved if every node of the mesh is considered as the centre of a cell. The accumulation matrix (M) is then handled node wise and no more element wise. The volume of the fictive cell replaces the spatial integration of the matrix M.

Mass lumping is widely used in the literature although many authors do not recommend it. In this work we seek for clearness in this question. In the first part a review on publications concerning mass lumping is presented. For comparison of both schemes, a numerical experiment is performed. Results using mass lumping and consistent integration for a practical two-dimensional transient problem are presented. In this example a contaminant leakage from a repository is simulated. The transport process through a porous medium is described by the advection-diffusion equation. Both diffusion dominated and advection dominated cases are analysed. The different approximation schemes for the accumulation matrix are compared in terms of accuracy and convergence rate for an iterative solver. The numerical efficiency is verified through comparison of the results with an analytical solution provided by Leij & Dane (1990). For the solution of the proposed problem we use the standard minimum residuum approach (Galerkin) and a stabilized scheme known as S^3 – Symmetrical Streamline Stabilization (Wendland & Schmid, 2000).

Literature Overview

The mass lumping procedure has been used through the years in different applications documented in the literature. This overview is divided in two tables, which present a summary of some previous works related to the use of mass lumping for the accumulation term. The first one (Table 1) presents the works favourable to mass lumping. The second one (Table 2) describes the works against mass lumping. As one will observe there is no a common agreement concerning this technique.

Although there are many publications favorable to the use of lumped mass matrix, there are also some works describing the poor behavior of this approach for the advection-diffusion equation. In order to form an own opinion on this question, the numerical solutions of both schemes will be compared for a 2-dimensional test case.

Test Problem

Governing equation

For the physical process proposed, we consider a single-phase (water) flow with a single dissolved solute moving through a homogeneous, saturated porous medium. The flow field is independent of the solute concentration and is at steady state. Chemical reactions and decay will not be considered, though they can easily be incorporated into the mathematical and numerical models.

In this case the transient transport of dissolved solutes is governed by the advection-dispersion equation (Bear, 1979).

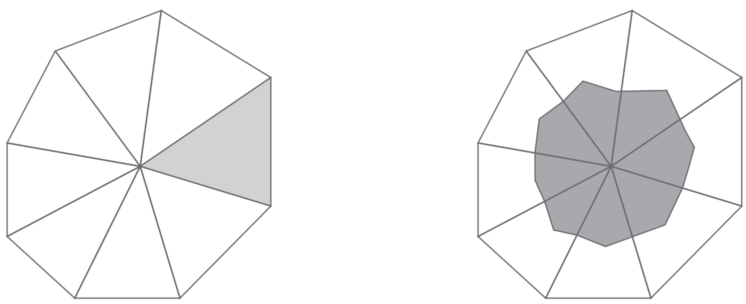

Figure 1 Physical interpretation of accumulation: a) consistent; b) lumping.

Table 1 Works favorable to mass lumping.

Authors	Year	Characteristics
Neumann	1973	Mass lumping for the unsaturated flow equation
Zienkiewicz & Morgan	1983	Stability analysis for explicit schemes shows that lumped mass allows larger Δt then the consistent one
Segerlind	1984	Mass lumping leads to stability in the solution, getting faster convergence with less oscillation
Celia *et al.*	1990	Stable solution for non-linear processes in the unsaturated zone could only be achieved through use of mass lumping
Helmig	1993	Experiences with the Lumped-Finite-Element formulation. The oscillation can indeed be reduced but the time-dependent solution quality is not necessarily improved
Eymard & Sonier	1994	Convergence proof for CVFE under grid constraints. CVFE faster then FD for petroleum reservoir simulation
Kung *et al.*	1994	CVFE for flexible-grid oil reservoir simulation
Pepper	1994	Application of compressible viscous flow computed with mass lumping
Franca & Russo	1996	Residual-free bubbles for the derivation of mass lumping
Giammarco *et al.*	1996	CVFE for the simulation of overland shallow flow. Results for traditional FEM with consistent mass matrix are not accurate enough
Elmkies & Joly	1997	Mass lumping for the 2D Maxwell equation
Oliveira	1997	Mass balance improvement through CVFE-ELM for flow and transport in estuaries
Franca *et al.*	1998	Derivation of mass lumping for FEM based on residual free bubbles. The scheme is applied for the advection-diffusion-reaction equation
Helmig & Huber	1998	Mass lumping for two-phase flow
Bastian & Helmig	1999	Application of mass lumping for the simulation of two-phase flow processes, associated with multigrid and parallel computation

Table 2 Works contrary to mass lumping.

Authors	Year	Characteristics
Gresho *et al.*	1978	Accuracy of the finite element method is reduced through lumping. The error increases with the Peclet number. For advection dominated processes the procedure becomes unstable. The technique still supplies acceptable results for pure diffusion
Huyakorn & Pinder	1983	Solution with the lumped matrix is less precise then with consistent integration
Sudicky	1989	Consistent matrix is more accurate for advection-diffusion
Gottardi & Venutelli	1997	Lmping increases the global error for advection-diffusion
Gottardi	1998	FEM better than CVFEL for 1D advection-diffusion
Hansko	1998	Conservation analysis in FEM concerning mass lumping
Christon & Voth	2000	RKPM applied for wave equation. Mass lumping leads to lagging phase errors in comparison to consistent matrix

$$\frac{\partial c}{\partial t} + \mathbf{v}\nabla c - \nabla(\mathbf{D}\nabla c) = 0$$

where *x* is the position vector, *t* is the time variable, *c(x,t)* is the solute concentration, v*(x)* is the fluid flow velocity, *D(x)* is the hydrodynamic dispersion tensor, consisting of the mechanical dispersion and molecular diffusion.

The mathematical model is completed by the definition of initial and boundary conditions, which can be of Dirichlet, Neumann and Cauchy type.

Test field

In order to compare both integration schemes for the mass matrix, a numerical experiment was performed. In this example a contaminant leakage from a repository is simulated. Both diffusion dominated and advection dominated cases are analysed. The different approximation schemes for the accumulation matrix are compared in terms of accuracy and convergence rate for an iterative solver.

The test domain consists on a rectangular section of 100 m length and 40 m width (Figure 2) with a contaminant source placed on the upstream boundary. The physical parameters for the porous medium have been chosen in order to provide a flow velocity of 1.0 m/s.

For the solute transport the boundary conditions consist of a specified concentration of c = 100% on the left side of the domain as shown in Figure 2. For the remaining boundaries homogeneous Neumann conditions (no solute flux over the boundary) has been chosen. The initial concentration was 0% elsewhere.

Results and Discussion

For the simulation of the considered problem the computer codes SICK100 (Schmid *et al*., 1991) and KASKADE (Erdmann *et al*., 1993) were employed. The problem was solved using regular meshes for different values of Peclet and Courant numbers up to a time of t = 50 s. The time discretization was full implicit (Euler scheme). The parameters used for the different cases are given in Table 3.

The legend shown in Figure 3 can be used for interpretation of the results shown in the following figures. The different shading represents the advance of the solute front from the injection line (c = 100%, gray) into the porous medium (c = 0%, white) as concentration isosurfaces.

In the first case the transport is diffusion dominated and of parabolic character. The stability criteria ($P_e < 2.0$ and $C_o < 1.0$) are observed. The numerical solutions (Figure 4) using consistent and lumped integration of the mass matrix for the Galerkin method are well behaved and agree with analytical results. The concentration distribution computed using mass lumping shows a higher cross-diffusion.

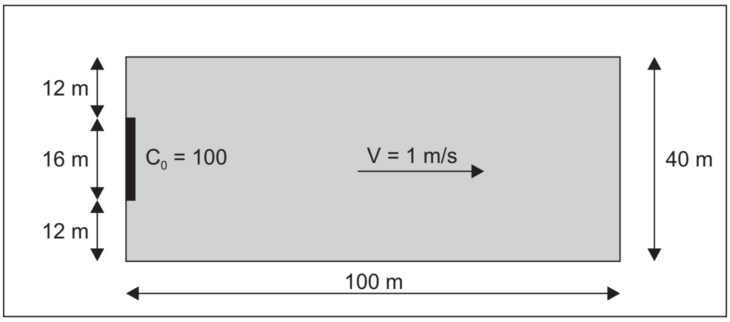

Figure 2 Geometry of the field-scale example.

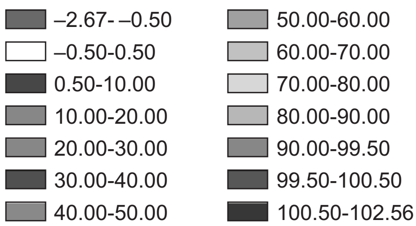

Figure 3 Legend for the concentration distributions shown in the following figures.

Table 3 Discretization parameters for the test example.

Case	$\Delta t(s)$	$D_L(m_2/s)$	$D_T(m_2/s)$	$\Delta x = \Delta y$ (m)	P_e	C_o
1	1.0	2.0	0.2	2.0	1.0	0.5
2	1.0	0.02	0.002	2.0	100.0	0.5
3	1.0	2.0	0.2	0.5	0.25	2.0
4	1.0	0.02	0.002	0.5	25.0	2.0

D_L = longitudinal dispersion. D_T = transversal dispersion.

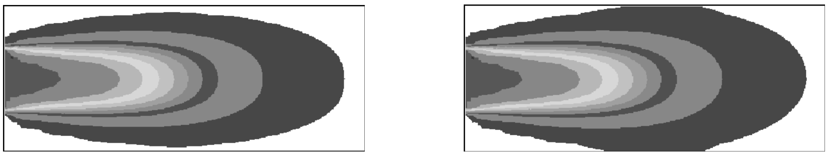

Figure 4 Concentration distribution for case 1 after 50 s.

In the second case the transport is advection dominated and of hyperbolic character. The stability criteria are not observed and difficulties due to numerical oscillation can be expected. The computed results are presented in Figure 5. Compared to the analytical solution, a higher longitudinal spreading and oscillation can be observed for both schemes. Furthermore one can recognize a stronger oscillation with negative concentrations ($c < 0\%$) for the solution using mass lumping. In the region of higher concentration overshooting ($c > 100\%$) can also be observed. The front of solute advance suffers on cross-diffusion and shows mesh dependency.

In order to reduce the oscillations and to get acceptable results, normally a fine discretization has to be chosen. The finite element mesh was also refined in x and y directions. For case 3 the results are equivalent to case 1 and will not be shown.

The results for case 4 shown in Figure 6 make evident the better results obtained through refinement. The transversal oscillation could be reduced. Also in this case the solution with mass lumping shows more oscillation than the consistent integration. The problem with cross-diffusion observed in the advancing front still remains.

Equivalent behavior was obtained using the S^3-scheme (with upwind) for the solution of the advection-diffusion equation for all cases studied.

According to the literature the main advantage of mass lumping in comparison with the consistent integration appears to be the saving on computational effort. In Table 4 a comparison of convergence behavior and computation time after 50 time steps for the different cases and solution schemes is shown. For the Galerkin scheme a BICG-STAB solver with SGS preconditioning was used. For the S^3-scheme (with upwind) a PCG solver could be used, viewing the symmetry of the system of equations.

Both the number of iterations for the iterative solver and the CPU time are smaller for the solution with mass lumping according to the observations in the literature. However the solution of the system of equations represents just a small part of the computation time in each time step. Due to the time necessary for matrix assemblage and post processing, the total CPU time consumed by the lumped integration is approximately the same of the consistent one. The advantage is also diminished considering all tasks necessary when using the finite element method.

Conclusion

The concept of mass lumping related to the finite element method as well as a review on pertinent works published in the literature has been compiled. Considering the different opinions associated with this technique a comparison of both schemes – consistent and lumped – was performed. The physical problem consists on the simulation of a contaminant leakage from a repository through a porous medium. In this case the transport process is described by the two-dimensional advection-diffusion equation. Both diffusion dominated and advection dominated cases were analysed.

The different approximation schemes for the accumulation matrix are compared in terms of accuracy and convergence rate for an iterative solver. The results obtained allow following conclusions for the case studied:

- Mass lumping leads to higher cross-diffusion.
- Mass lumping leads to unphysical oscillation in the results.
- Mass lumping accelerates the convergence of the solution (iterative solver).
- For the case studied, mass lumping does not show any significant advantage.

Considering these statements, the use of mass lumping for the advection-diffusion in situations similar to the one analysed in this work should be avoided.

Table 4 Computational data for the test example.

Case	**1 (diffusive)**	**2 (advective)**	**3 (diffusive)**	**4 (advective)**
	(1071 dof, 2000 ele)		**(4141 dof, 8000 ele)**	
Iterations solver/Δt				
G-C	25	40	32	40
G-L	17	20	27	14
S^3-C	10	10	18	16
S^3-L	9	5	19	10
CPU total (s)				
G-C	204.44	198.17	2872.76	2684.95
G-L	203.00	194.90	2914.21	2678.48
S^3-C	5.61	5.61	23.76	23.23
S^3-L	5.63	5.48	23.84	22.58

G-C = Galerkin consistent; G-L = Galerkin lumped; S^3-C = Upwind consistent; S^3-L = Upwind lumped; dof = degrees of freedom; ele = number of elements.

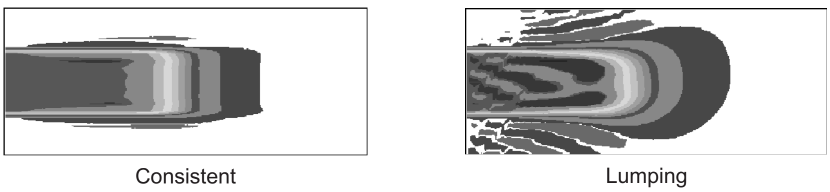

Figure 5 Concentration distribution for case 2 after 50 s.

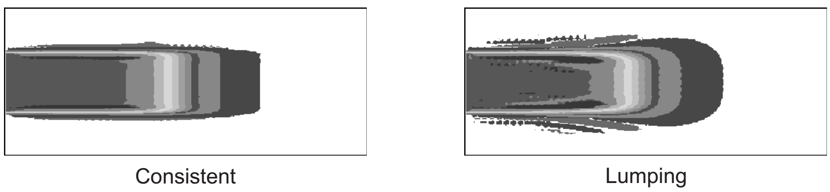

Figure 6 Concentration distribution for case 4 after 50 s.

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